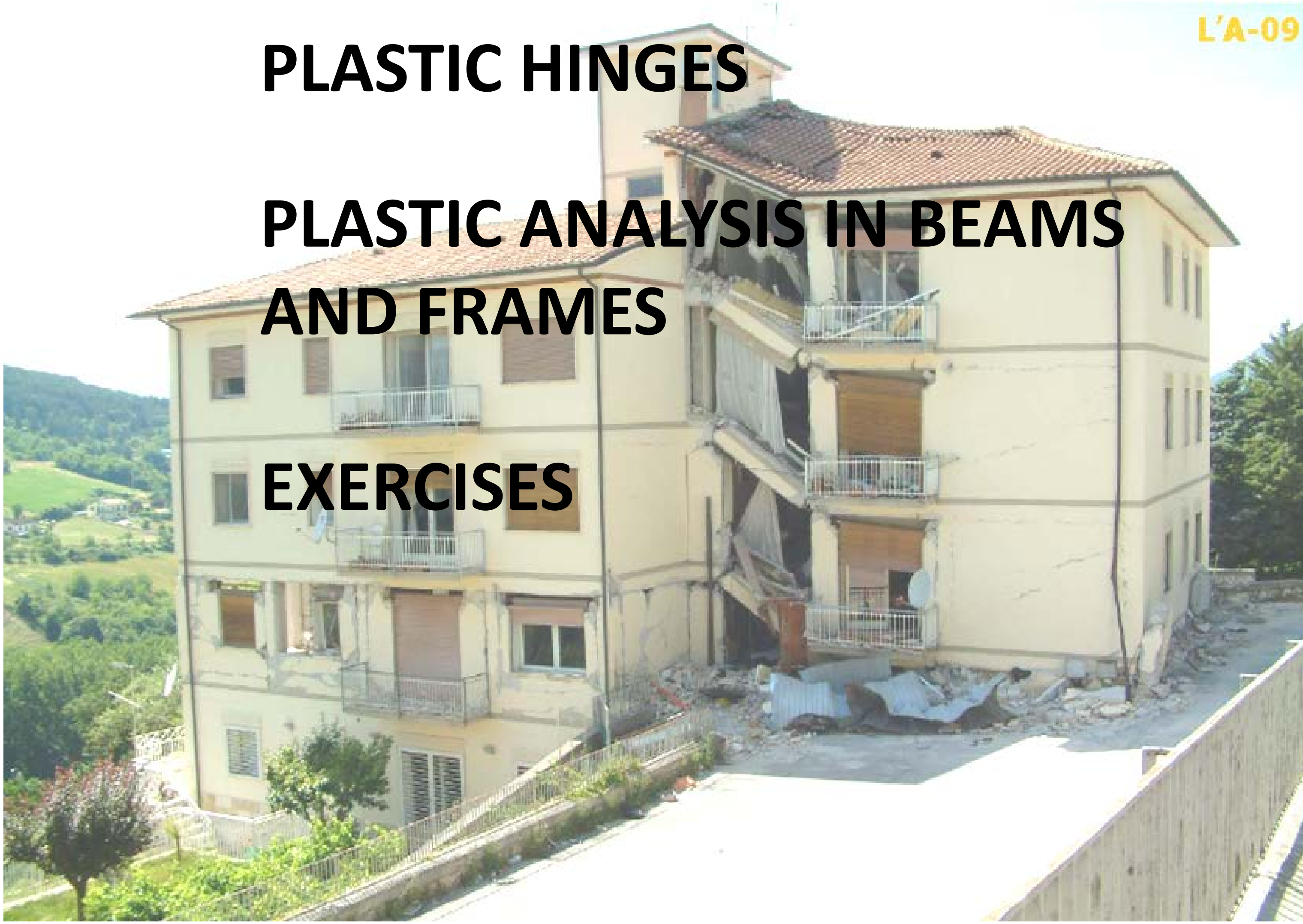
RESISTÈNCIA DE MATERIALS I ESTRUCTURES A L'ENGINYERIA
UNIVERSITAT POLITÈCNICA DE CATALUNYA

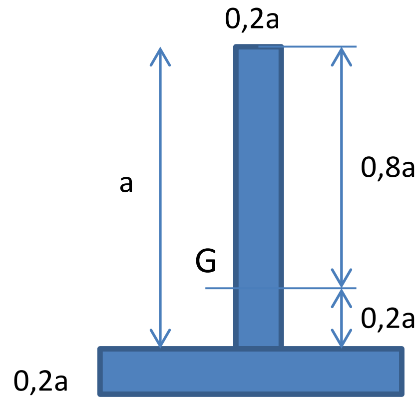
PLASTIC HINGES

PLASTIC ANALYSIS IN BEAMS AND FRAMES

EXERCISES



etseccpb
Escola Tècnica Superior d'Enginyers de Camins, Canals i Ports



Gravity center (Static moments)

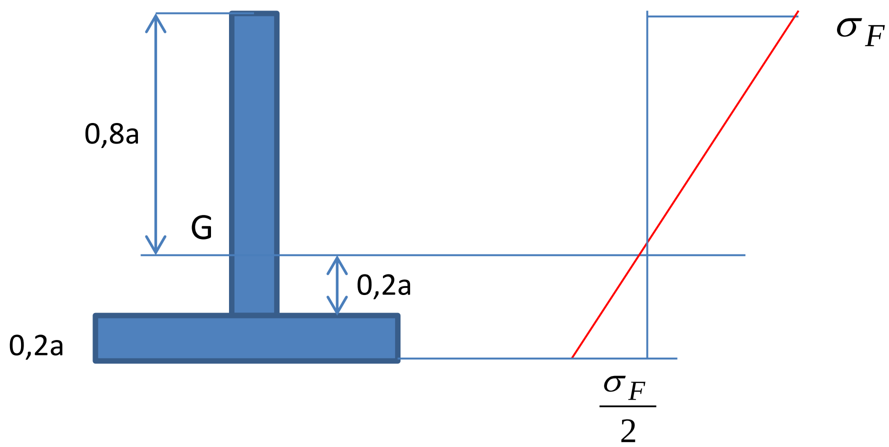
$$a \times 0,2\,a \times \left(\frac{a}{2} + 0,2\,a\right) + a \times 0,2\,a \times \frac{0,2\,a}{2} = Y_G \times 2 \times a \times 0,2\,a \qquad Y_G = 0,4\,a$$

Moment of Inertia (Steiner Theorem)

$$I = \frac{1}{12} \times 0,2\,a \times a^3 + a \times 0,2\,a \times (0,5a - 0,2a)^2 + \frac{1}{12} \times a \times (0,2\,a)^3 + a \times 0,2\,a \times (0,4a - 0,1a)^2$$

$$I = \frac{160}{3} \times 10^{-3}\,a^4$$

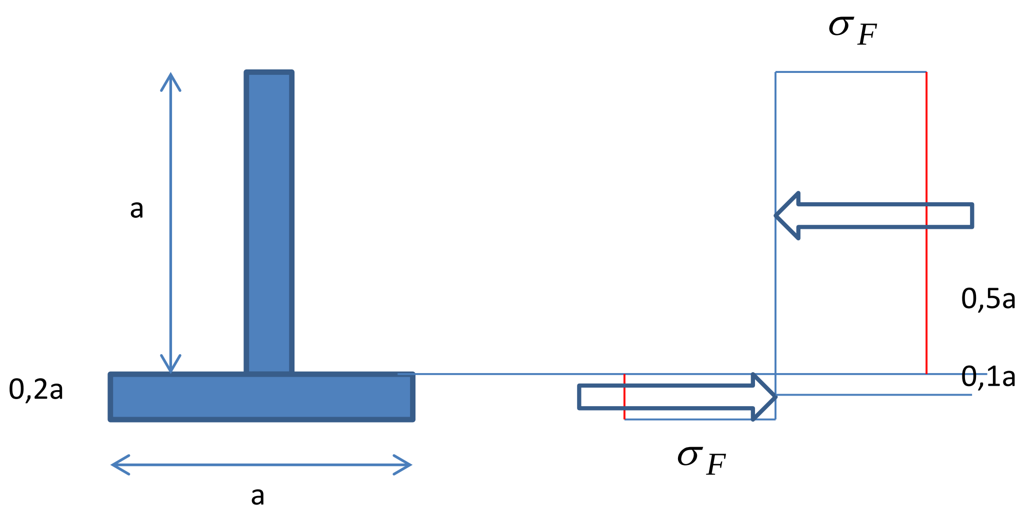
Elastic Limit, Bending Moment



$$\sigma = \frac{M\,y}{I} \qquad M = \frac{\sigma\,I}{y}$$

$$M_{\ell e} = \frac{160\times10^{-3}a^4}{3\times0{,}8\,a}\sigma_F = \frac{20}{3}\times10^{-2}a^3\sigma_F$$

Plastic Bending Moment

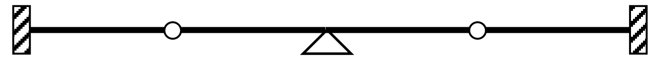


$$M_{\ell p} = a\times0{,}2\,a\times0{,}6\,a\times\sigma_F =$$

$$1{,}2\times10^{-2}a^3\,\sigma_F$$

$$f = 1{,}8$$

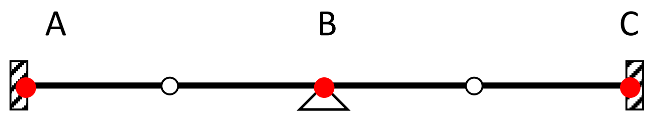
For the beam in the figure:

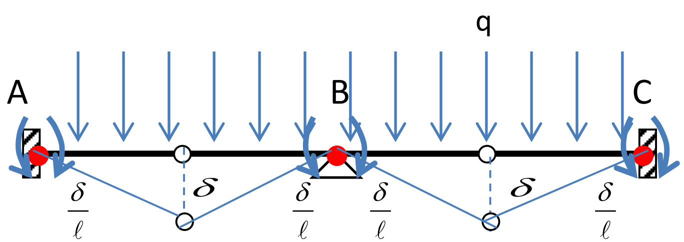


Compute the limit uniform load for a limit bending moment of value M

Calculate the static reaction forces and draw the bending moment law for M_f

The beam has 2 intermediate hinges and the 4 sections are of the same length l



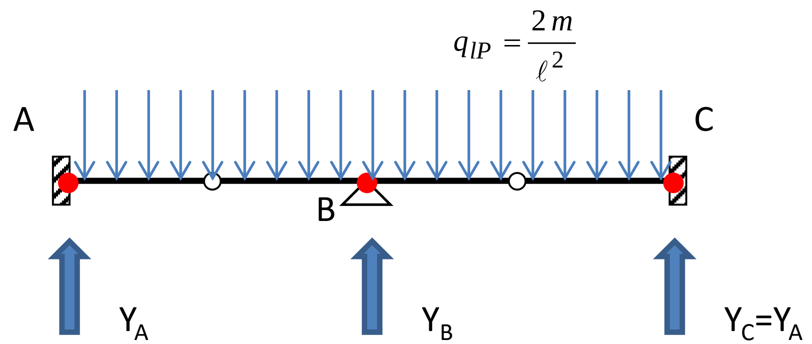


Symmetry

Single mechanism

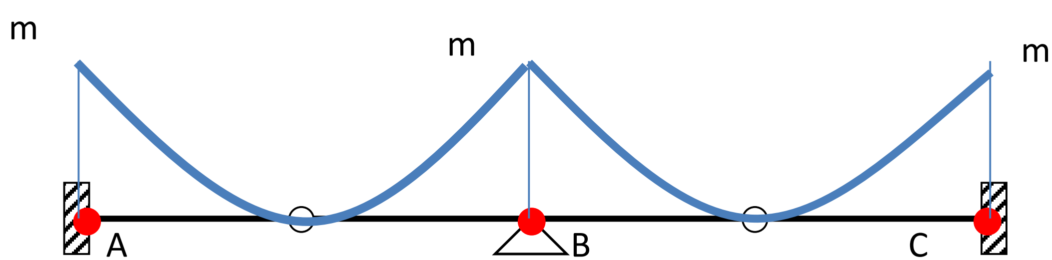
$$q\times\frac{1}{2}2\,\ell\times\delta+q\times\frac{1}{2}2\,\ell\times\delta=-M_A\times\frac{\delta}{\ell}-M_B\times\left(\frac{\delta}{\ell}+\frac{\delta}{\ell}\right)-M_C\,\frac{\delta}{\ell}$$

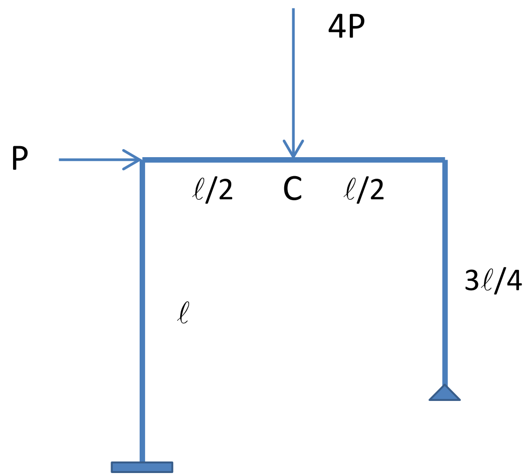
$$q_{lP}=\frac{2\,m}{\ell^2}$$



$$Y_A \times \ell - \frac{2m}{\ell^2} \times \ell \times \frac{\ell}{2} - m = 0 \qquad Y_A = Y_C = \frac{2m}{\ell}$$

$$Y_B = \frac{2m}{\ell^2} \times 4\ell - 2 \times \frac{2m}{\ell} = \frac{4m}{\ell}$$



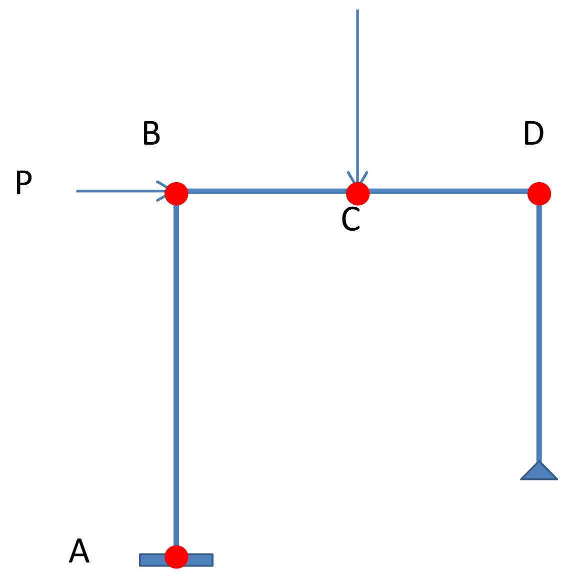


Critical Sections A B C D

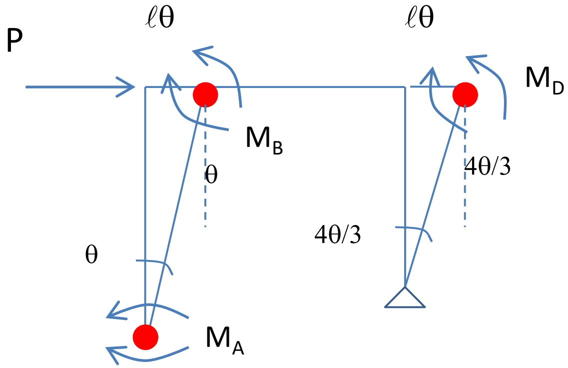
Hyperestatic degree 2

Plastic Hinges 3

Mechanisms 4



Translational Mechanism

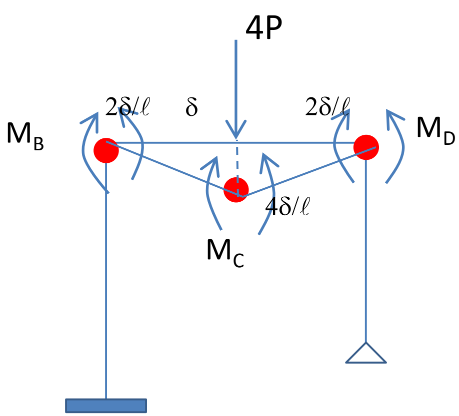


$$-M_A = M_B = -M_D = M_P$$

$$P \times \vartheta \ell = -M_A \times \vartheta + M_B \times \vartheta - M_D \times \frac{4\vartheta}{3}$$

$$P\ell = \frac{10 M_P}{3} \qquad P_{lp} = \frac{10 M_P}{3\ell}$$

Lintel Failure Mechanism



$$-M_B = M_C = -M_D = M_P$$

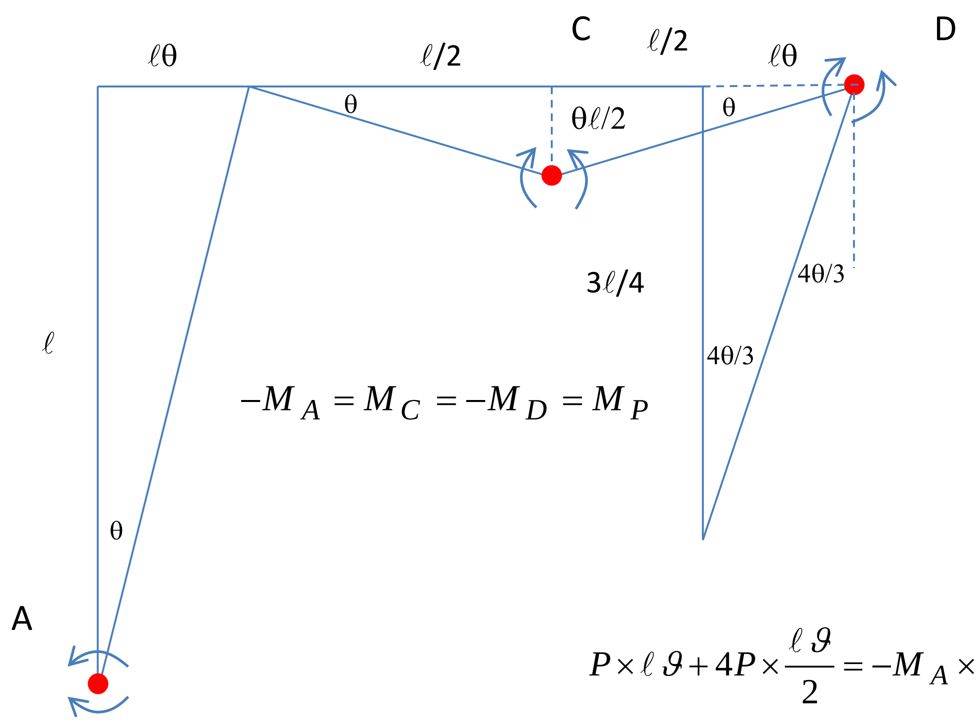
$$4P \times \delta = -M_B \times \frac{2\delta}{\ell} + M_C \times \left(\frac{2\delta}{\ell} + \frac{2\delta}{\ell}\right) - M_D \times \frac{2\delta}{\ell}$$

$$4P = \frac{8 M_P}{\ell} \qquad P_{lp} = \frac{2 M_P}{\ell}$$

Combination of Mechanisms

MECHANISM	A	B	C	D	T_e x Pl	T_i x M_p/l	P_{lp} x M_p/l
ABD	-1	+1	0	-4/3	1	10/3	10/3
BCD	0	-2	4	-2	4	8	2
ACD 2 x ABD + BCD	-2	0	4	-14/3	6	32/3	32/18 <2

Mechanism ACD

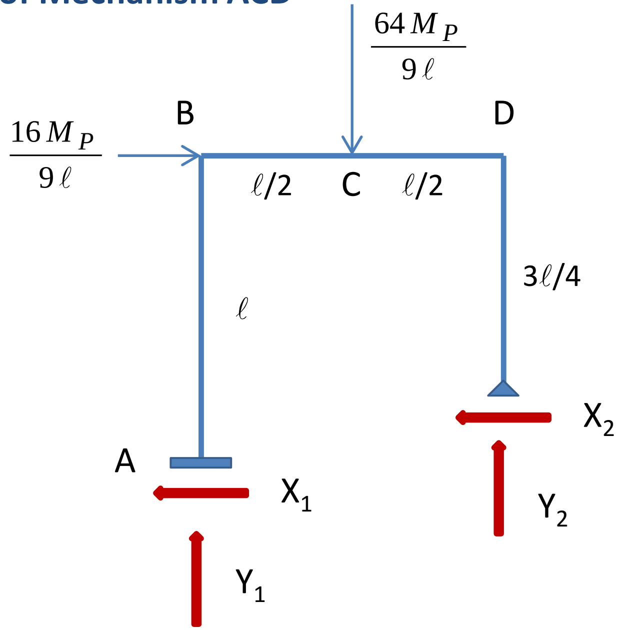


$$-M_A = M_C = -M_D = M_P$$

$$P \times \ell\,\vartheta + 4P \times \frac{\ell\,\vartheta}{2} = -M_A \times \vartheta + M_C \times (\vartheta + \vartheta) - M_D \times \left(\vartheta + \frac{4\,\vartheta}{3}\right)$$

$$3P = \frac{8\,M_P}{\ell} \qquad P_{lp} = \frac{16\,M_P}{9\,\ell}$$

Verification of Mechanism ACD



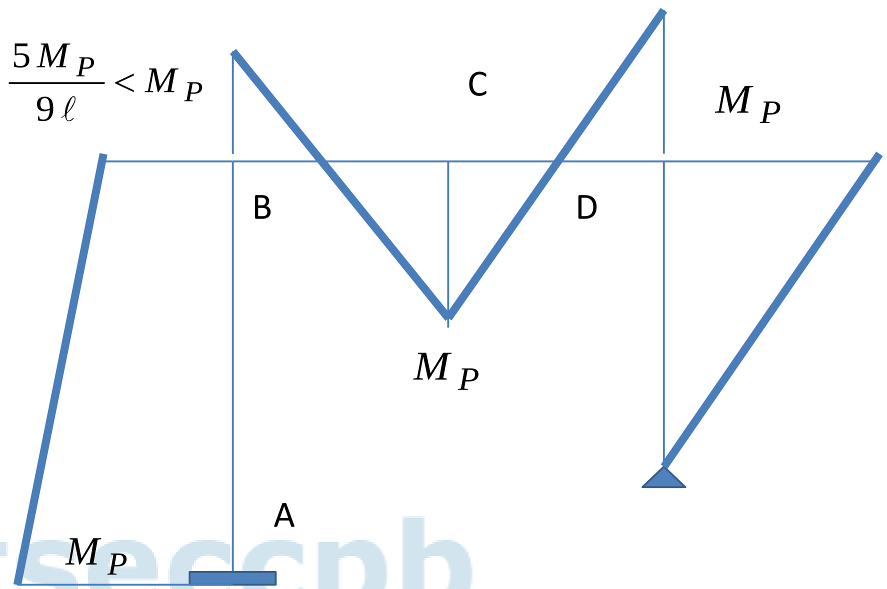
$$X_2 \times \frac{3\ell}{4} = M_P \qquad X_2 = \frac{4M_P}{3\ell}$$

$$X_1 = \frac{16M_P}{9\ell} - \frac{4M_P}{3\ell} = \frac{4M_P}{9\ell}$$

$$Y_2 \times \frac{\ell}{2} - M_P = M_P \qquad Y_2 = \frac{4M_P}{\ell}$$

$$Y_1 = \frac{64M_P}{9\ell} - \frac{4M_P}{\ell} = \frac{28M_P}{9\ell}$$

$$M_B = X_1 \times \ell - \frac{4M_P}{9\ell} - M_P = -\frac{5M_P}{9\ell} < -M_P$$

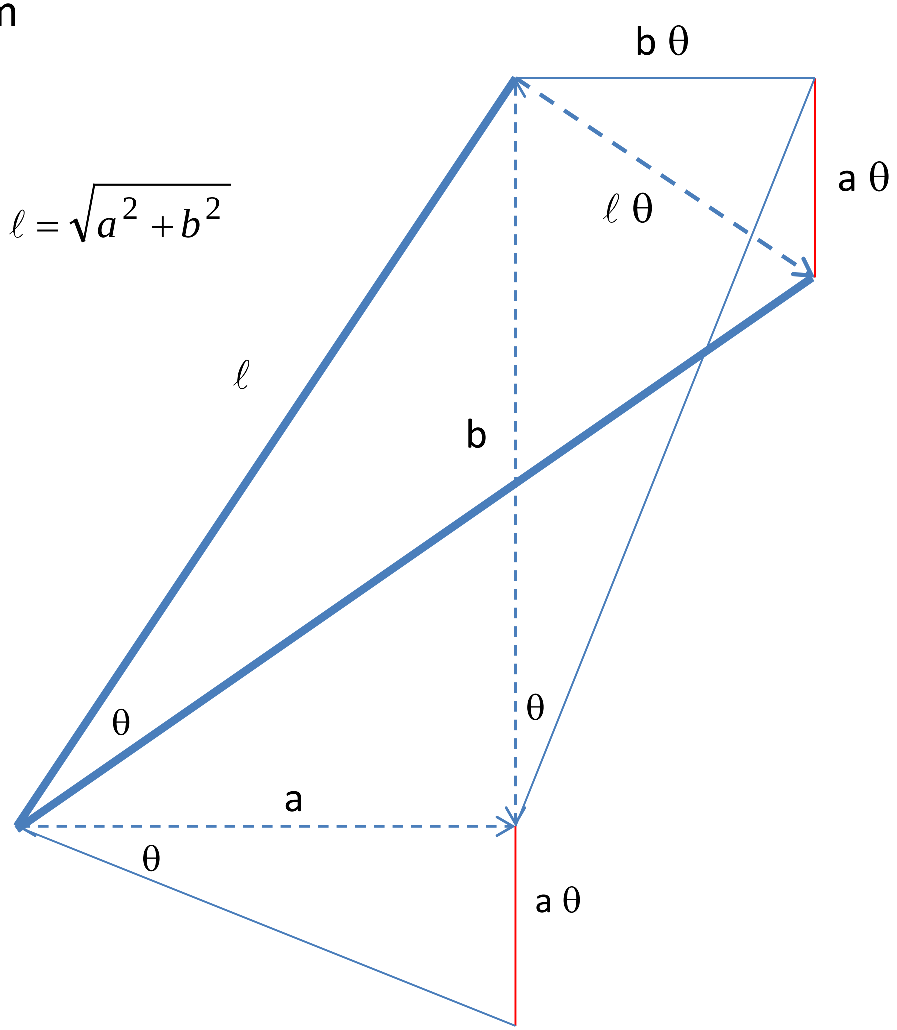


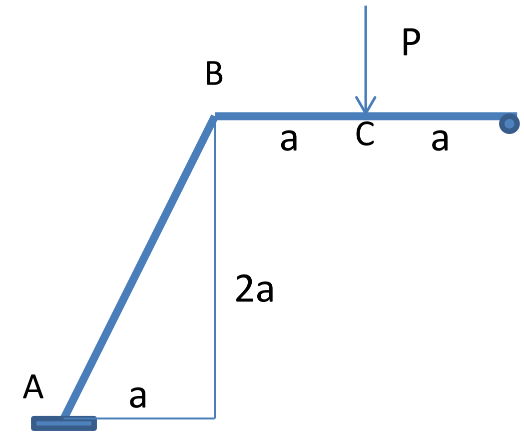
The M_P is not exceeded

$$P_{lp} = \frac{16M_P}{9\ell}$$

Movement of the leaning beam

$$\ell = \sqrt{a^2 + b^2}$$





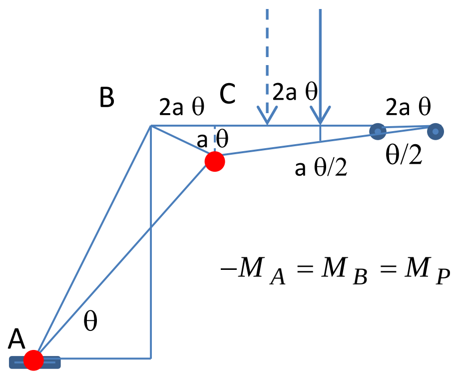
Critical Sections A B C

Hyperestatic degree 1

Plastic Hinges 2

Mechanisms 3

Translational Mechanism

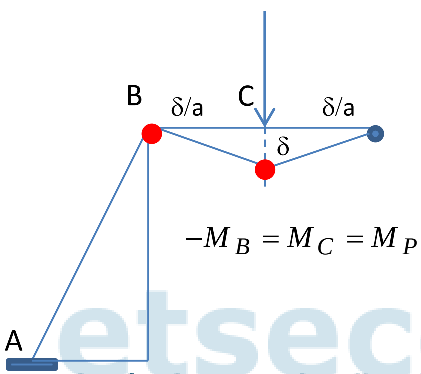


$$-M_A = M_B = M_P$$

$$P \times a\frac{\vartheta}{2} = -M_A \times \vartheta + M_B \times \left(\vartheta + \frac{\vartheta}{2}\right)$$

$$P\,a = -2\,M_A + 3\,M_B \qquad P_{lp} = \frac{5\,M_P}{a}$$

Lintel Failure Mechanism



$$-M_B = M_C = M_P$$

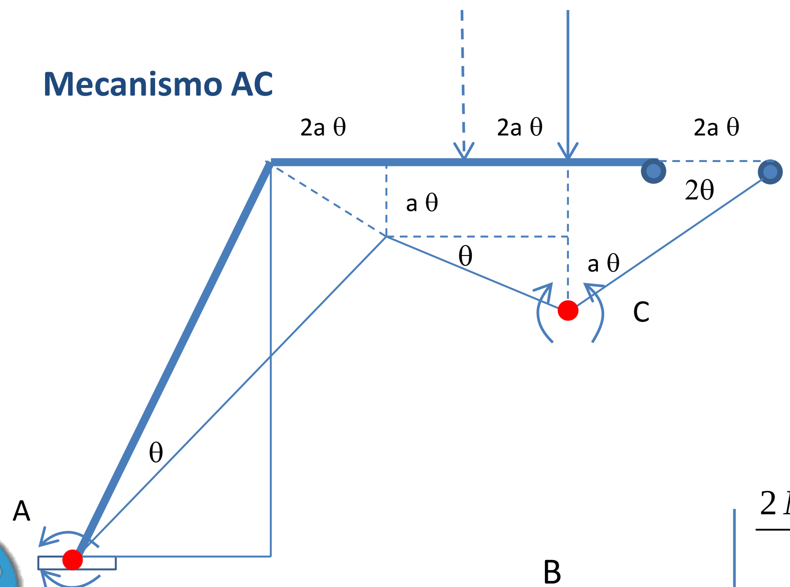
$$P \times \delta = -M_B \times \frac{\delta}{a} + M_C \times \left(\frac{\delta}{a} + \frac{\delta}{a}\right)$$

$$P\,a = -M_B + 2\,M_C \qquad P_{lp} = \frac{3\,M_P}{a}$$

Mechanism Combination

MECHANISM	A	B	C	T_e x Pa	T_i x M_p/a	P_{lp} x M_p/a
AB	-2	3	0	1	5	5
BC	0	-1	2	1	3	3
AC ABD +3 x BC	-2	0	6	4	8	2

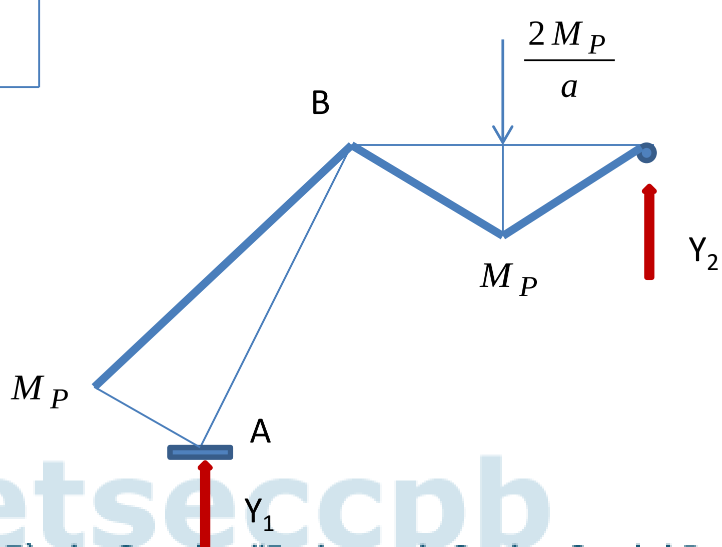
Mecanismo AC



$$P \times 2a\vartheta = -M_A \times \vartheta + M_C \times (\vartheta + 2\vartheta)$$

$$P \times 2a = -M_A + 3M_C \qquad P_{lp} = \frac{2M_P}{a}$$

Verification of Mechanism AC

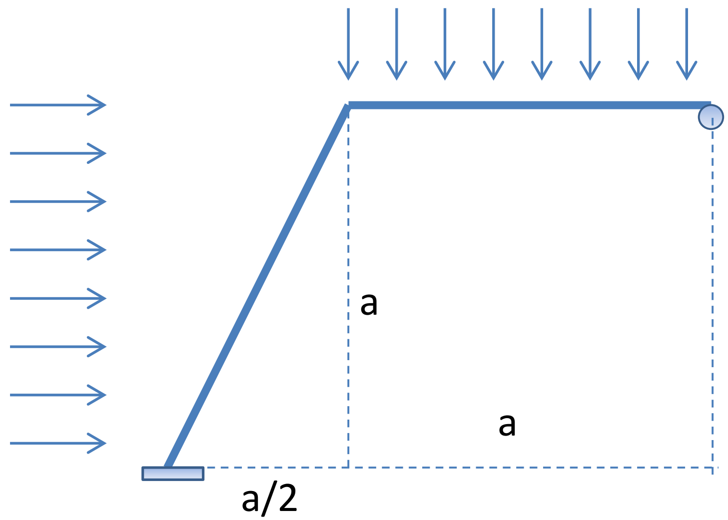


$$Y_2 \times a = M_P \qquad Y_2 = \frac{M_P}{a} \qquad Y_1 = \frac{M_P}{a}$$

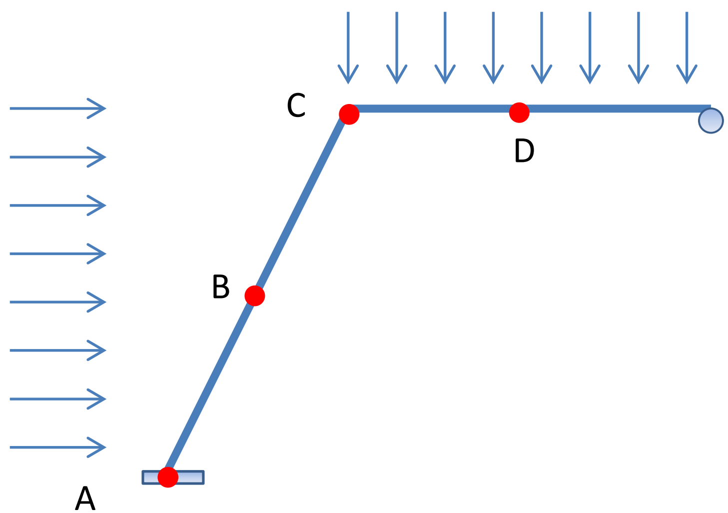
$$M_B = Y_2 \times 2a - \frac{2M_P}{a} \times a = 0 < M_P$$

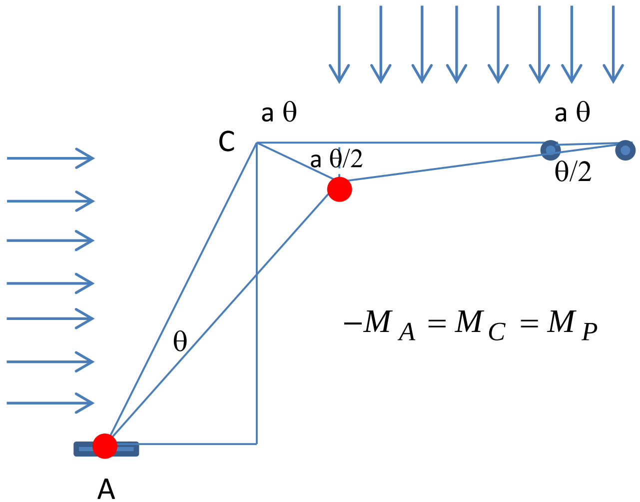
$$M_B = Y_1 \times a - M_P = 0 < M_P$$

M_P is not exceeded



Critical Sections	4
Hyperestatic Degree	1
Plastic Hinges	2
Mechanisms	3



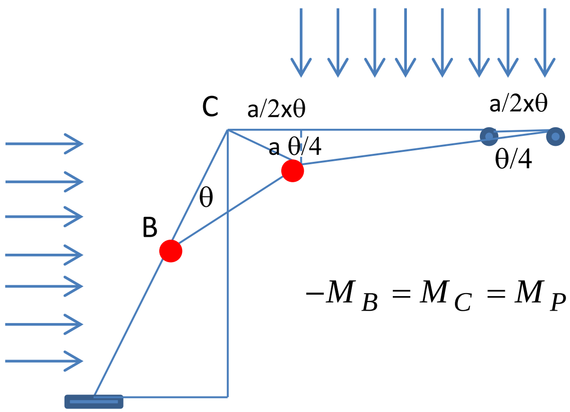


$$-M_A=M_C=M_P$$

Translational Mechanism

$$q\times\frac{1}{2}a\times a\,\vartheta+q\frac{1}{2}a\times a\frac{\vartheta}{2}=-M_A\times\vartheta+M_C\times\left(\vartheta+\frac{\vartheta}{2}\right)$$

$$3q\times a^2=-4M_A+6M_C \qquad q_{lp}=\frac{10M_P}{3a^2}$$

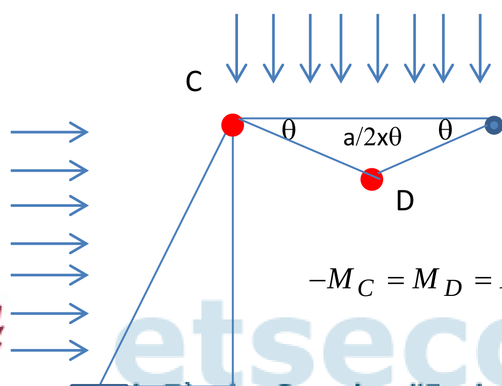


$$-M_B=M_C=M_P$$

Pillar failure mechanism

$$q\times\frac{1}{2}\frac{a}{2}\times\frac{a}{2}\vartheta+q\frac{1}{2}a\times a\frac{\vartheta}{4}=-M_B\times\vartheta+M_C\times\left(\vartheta+\frac{\vartheta}{2}\right)$$

$$q\times a^2=-4M_B+6M_C \qquad q_{lp}=\frac{10M_P}{a^2}$$



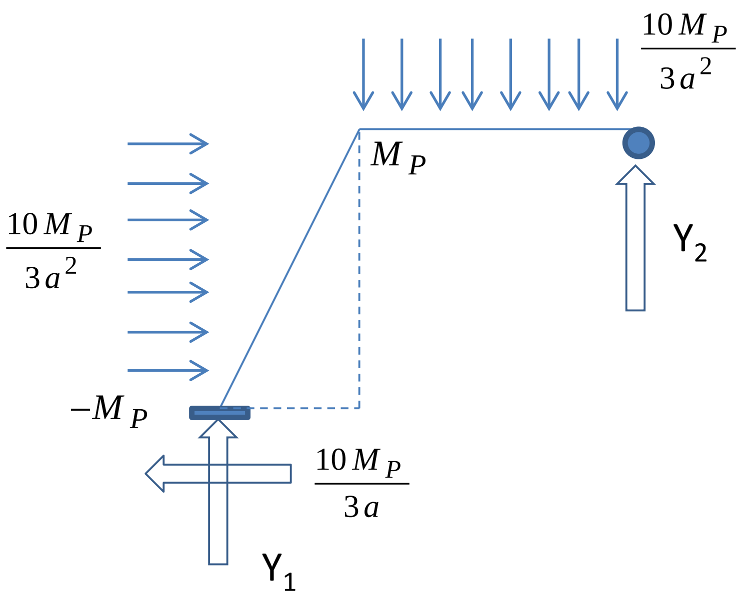
$$-M_C=M_D=M_P$$

Lintel Failure Mechanism

$$q\times\frac{1}{2}a\times\frac{a}{2}\vartheta=-M_C\times\vartheta+M_D\times(\vartheta+\vartheta)$$

$$q\times a^2=-4M_C+8M_D \qquad q_{lp}=\frac{12M_P}{a^2}$$

Verification of Mechanism AC



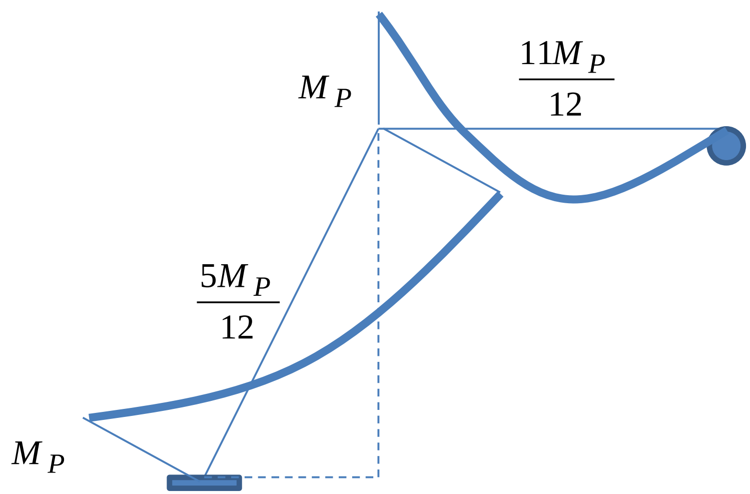
$$Y_2 \times a - \frac{10\,M_P}{3a^2} \times a \times \frac{a}{2} = M_P$$

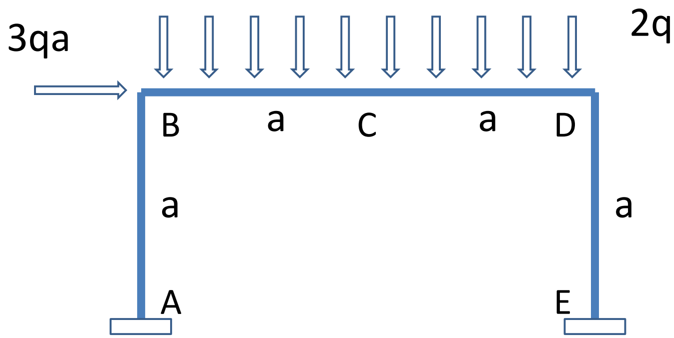
$$Y_2 = \frac{8\,M_P}{3a} \qquad Y_1 = \frac{2\,M_P}{3a}$$

$$M_D = \frac{8\,M_P}{3a} \times \frac{a}{2} - \frac{10\,M_P}{3a} \times \frac{a}{2} \times \frac{a}{4} = \frac{11}{12} M_P < M_P$$

$$M_B = \frac{10\,M_P}{3a} \times \frac{a}{2} + \frac{2\,M_P}{3a} \times \frac{a}{4} - \frac{10\,M_P}{3a^2} \times \frac{a}{2} \times \frac{a}{4} - M_P = \frac{5\,M_P}{12} < M_P$$

M_P is not exceeded





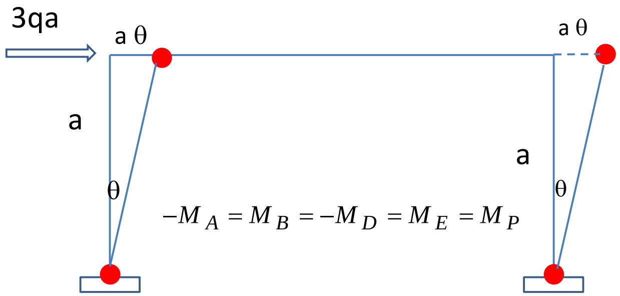
Critical Sections A B C D E

Hyperestatic degree 3

Plastic Hinges 4

Mechanism 3

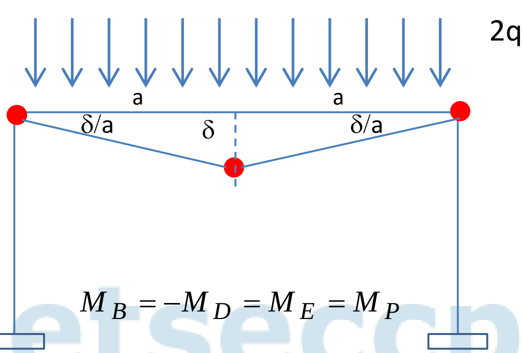
Translational Mechanism



$$3qa \times a\vartheta = -M_A \times \vartheta + M_B \times \vartheta - M_D \times \vartheta + M_E \times \vartheta$$

$$3qa^2 = -M_A + M_B - M_C + M_D \qquad q_{lp} = \frac{4M_P}{3a^2}$$

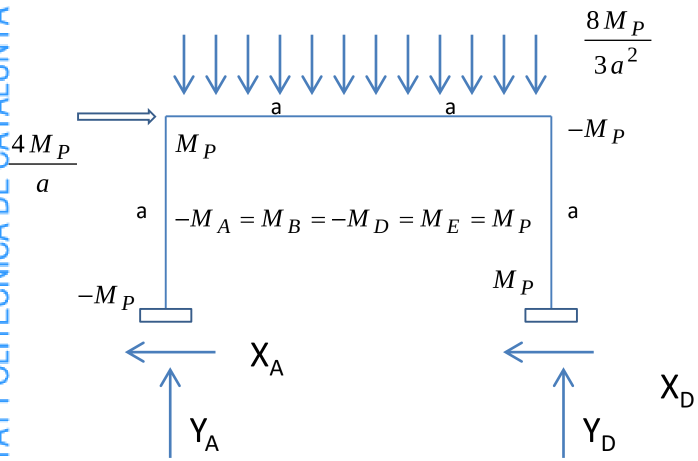
Lintel Failure Mechanism



$$2q\frac{1}{2}2a \times \delta = -M_B \times \frac{\delta}{a} + M_C \times \left(\frac{\delta}{a} + \frac{\delta}{a}\right) - M_D \times \frac{\delta}{a}$$

$$2qa^2 = -M_B + 2M_C - M_D \qquad q_{lp} = \frac{2M_P}{a^2}$$

Verification of Mechanism ABDE



$$-X_D \times a + M_P = -M_P \qquad X_D = \frac{2M_P}{a} \qquad X_A == \frac{2M_P}{a}$$

$$Y_D \times 2a - \frac{2M_P}{a} \times a + M_P - \frac{8M_P}{3a^2} \times 2a \times a = M_P$$

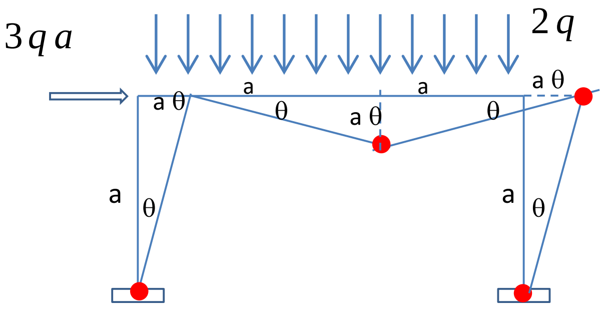
$$Y_D = \frac{11M_P}{3a} \qquad Y_A = \frac{8M_P}{3a^2} \times 2a - \frac{11M_P}{3a} = \frac{5M_P}{3a}$$

$$M_C = \frac{11M_P}{3a} \times 2a - \frac{2M_P}{a} \times a + M_P - \frac{8M_P}{3a^2} \times a \times \frac{a}{2} = 5M_P > M_P$$

M_P is exceeded

Verification of Mechanisms

MECHANISM	A	B	C	D	E	T_e x qa^2	T_i x M_p/l	P_{lp} x M_p/a^2
ABDE	-1	+1	0	-1	+1	3	4	1,33
BCD	0	-1	2	-1	0	2	4	2
ACDE ABDE + BCD	-1	0	2	-2	+1	5	6	1,2

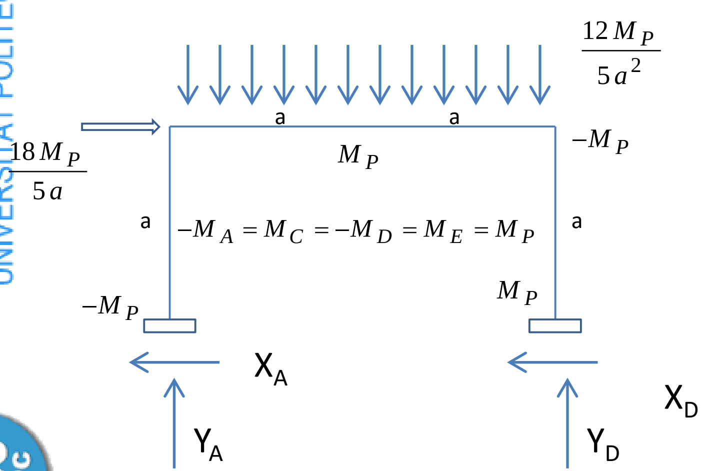


$$-M_A = M_C = -M_D = M_E = M_P$$

$$3qa\times a\vartheta + 2q\times\frac{1}{2}2a\times a\vartheta = -M_A\times\vartheta + M_C\times(\vartheta+\vartheta) - M_D\times(\vartheta+\vartheta) + M_E\times\vartheta$$

$$5qa^2 = -M_A + 2M_B - 2M_C + M_D \qquad q_{lp} = \frac{6M_P}{5a^2} < \frac{4M_P}{3a^2} < \frac{2M_P}{a^2}$$

Verification of Mechanism ACDE



$$-X_D\times a + M_P = -M_P \qquad X_D = \frac{2M_P}{a} \qquad X_A == \frac{8M_P}{5a}$$

$$M_B = \frac{8M_P}{5a}\times a - M_P = \frac{3M_P}{5a} < M_P$$

M_P is not exceeded